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Post-Newtonian Equations for the Metric Perturbation Generated by a Rotating Elastic Earth

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By using Synge's approximation method to describe the unperturbed problem, we obtain the equations for the gravitational field perturbation and the Lagrangian displacement that occur when an isolated and initially self-gravitating spherical and static elastic earth gets into steady rotation. These equations are explicitly derived in an order of approximation for the initial problem where both rotation and elastic structure manifest themselves in the perturbed state.

1. INTRODUCTION

If in the classical theory of elasticity an earth initially behaves as an elastic sphere, in static equilibrium, and under such a strong self-gravitational action that the theory of superposable small strains cannot be applied, it is still possible to study naturally its figure of equilibrium when it is in steady rotation by assuming its inner structure to be of homogeneous and incompressible type. The reason for this possibility rests upon the fact that these two assumptions allow us to consider the initial stress as being of hydrostatic type so that a Hook stress tensor for the assumed elastic perturbation may be added to the initial stress in order to describe the new perturbed state (see, e.g., Love, 1944).

But when, following the same procedure, the analogous problem is attempted to be formulated in general relativity, then two essential difficulties arise: First, it is impossible to define the figure of the earth until the gravitational problem has already been solved (which, in turn, requires the

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energy distribution to be previously specified), and second, an incompatibility between relativistic principles and the assumption of incompressibility becomes apparent, as this assumption leads to a speed of sound higher than the speed of light.

To avoid these difficulties, two different approaches, one in terms of exact formulations and the other by using post-Newtonian approximations (therefore, each one with its own advantages and limitations) have been carried out in the past. Thus, by establishing suitable generalizations of Hook's law, general relativistic exact formulations of elastic bodies subject to initial stress have been carried out by Glass and Winicour (1973) and Carter and Quintana (1972). Carter and Quintana (1975) and Quintana (1976) have also applied their generalization to study deformations of compact bodies by means of models in which a Schwarzschild space-time geometry is typically associated to them in their initial state.

By means of the second approach, that is to say, by assuming that a weak gravitational field is generated by the material system under consideration, more general models can be studied and approximate solutions for Einstein's equations can be obtained. Following this approach, Chandrasekhar and Esposito (1970), Anderson and Decanio (1975), and Ehlers (1980) have developed models for material systems when these are of hydrodynamic type.

Now, as Synge's approximation method (Synge, 1970) allows a direct analysis of the behavior of material systems with more general inner structures than those of hydrodynamic type (such as bodies with elastic structure), it is clearly well adapted to study the relativistic problem analogous to the classic one described above. Then, using this method (therefore, following the second approach), and taking into account the methodology of the first one, we derive in this paper the equations for the gravitational field perturbation and the Lagrangian displacement that occur when an initially spherical and static earth, prestressed by its own gravitational attraction, gets into steady rotation.

In connection with the first approach, it is of interest to mention again that in this approach both the cited initial configuration and the energy-momentum tensor are provisional descriptions in a flat background space-time to be used only in describing the initial state. Therefore, following this methodology, once the solution for the gravitational problem is derived in a predetermined order of approximation, the intrinsic geometry for the earth's surface can be specified by means of invariant measures.

The sequence of the paper is as follows: In Section 2 the geometry of the material system is defined and the basic principles of the relativistic theory of elasticity for a prestressed body that are needed for this work are briefly summarized. As has been said, to this end we follow the line traced by Glass and Winicour (1973) and Carter and Quintana (1972). Coordinate-free notation is used to avoid confusion with the coordinate expressions that appear in the following sections. Then, by imposing coordinate symmetries compatible with an axis-symmetric earth in steady rotation, in Section 3 general equations, that is to say, the equations corresponding to an unperturbed problem solved in an arbitrary order of approximation and showing the metric perturbation and the Lagrangian displacement, are derived. Next, using these equations, the corresponding one for an earth having in the perturbed state an isotropic and elastic structure, being prestressed by its own gravitational action, and behaving as a perfect fluid in the original unperturbed state are derived in Section 4. This derivation is carried out in the lowest order of approximation where the elastic structure manifests itself. In an Appendix some formulas needed to obtain these equations are given.

2. DESCRIPTION OF THE ELASTIC SOLID

2.1. Deformation and Stress

Let $\mathscr{S}$ be a four-dimensional differentiable manifold. A continuum body is defined in it as an open subset $\mathscr{D} \subset \mathscr{S}$. On the 4-manifold M of the class $\mathscr{C}^\infty$, connected, of Hausdorff type, and possessing a metric tensor field $\mathbf{g}$ of type $\binom{0}{2}$ and signature $(+,+,+,-)$, the motion of $\mathscr{D}$ is determined by the world tube defined by the one-to-one application

$$\phi:\quad \mathscr{D}\times\mathbb{R}\to M \tag{1}$$

where $\mathbb{R}$ represents the set of real numbers. The application $\phi_X:\mathbb{R}\to M$ defined by $\phi_X(t):=\phi(X,t)$ is called a world line. The 4-velocity field corresponding to any world tube is given by the application from M onto the tangent fiber bundle TM:

$$\begin{aligned}\mathbf{u}:\quad & M\to TM\\ & x\to\frac{\partial\phi}{\partial t}(X,t)\end{aligned} \tag{2}$$

where $\phi(X,t)=x$ and

$$\mathbf{u}^b\cdot\mathbf{u}=-1 \tag{3}$$

Here the superscript b represents the operator that, when acting onto any tensor, results in its associated covariant tensor, and the dot represents the obvious contraction. In more general cases, the contraction of the i_lth contravariant with the j_lth covariant index will be written as $C^{\dots il\dots}_{\dots jl\dots}$.

The physical properties of the material system are characterized by the symmetric tensor field of type $\binom{2}{0}$. $\mathbf{T}$ (the energy-momentum tensor), defined by

$$\mathbf{T} := \rho \mathbf{u} \otimes \mathbf{u} + \mathbf{S} \tag{4}$$

where ρ denotes the energy density and $\mathbf{S}$ is the symmetric tensor field of type $\binom{2}{0}$ called the stress tensor. $\mathbf{S}$ satisfies the orthogonality condition

$$\mathbf{S} \cdot \mathbf{u}^b = 0 \tag{5}$$

and is related to ρ by means of the characteristic equation of state for the material system.

Using the unit tensor δ and the 4-velocity (2), the orthogonal project tensor field of type $\binom{1}{1}$, which we shall denote by $\mathscr{P}$, is defined by

$$\mathscr{P} := \delta - \mathbf{u} \otimes \mathbf{u} \tag{6}$$

From this projection operator, $\mathscr{P}^b$ can be formed to provide a metric onto the vector subspace $S_x \subset T_x M$ orthogonal to $\mathbf{u}$ at the point $x \in M$. Therefore, $\mathscr{P}^b$ constitutes the generalization of the right Cauchy-Green deformation tensor of the classic theory of elasticity.

At every point of the manifold M the orthogonal projector (6) satisfies the conditions

$$\mathscr{P} \cdot \mathbf{u} = 0 \tag{7}$$

and

$$\mathscr{P} \cdot \mathscr{P} = \mathscr{P} \tag{8}$$

If now, following Rayner (1963), the auxiliary metric $\underset{\circ}{\mathscr{P}}^b$ onto S_x is considered, as this metric determines the distance between close world lines in the equilibrium state and satisfies both the orthogonality condition

$$\underset{\circ}{\mathscr{P}}^b \cdot \mathbf{u} = 0 \tag{9}$$

and the rigid motion condition in the sense of Born, that is to say, that the Lie derivative of $\underset{\circ}{\mathscr{P}}^b$ with respect to $\mathbf{u}$ vanishes,

$$\mathscr{L}_{\mathbf{u}} \underset{\circ}{\mathscr{P}}^b = 0 \tag{10}$$

then a symmetric tensor field of type $\binom{0}{2}$ (the relative strain tensor) can be defined:

$$\mathbf{e} := \tfrac{1}{2}(\mathscr{P}^b - \underset{\circ}{\mathscr{P}}^b) \tag{11}$$

2.2. Perturbations

Let us now assume that a perturbation is produced on the material system so that the initial space-time $(M, \mathbf{g})$ is transformed into another

$(M(1), \mathbf{g}(1))$. Furthermore, let us assume that there exists a one-parameter family of space-times $(M(\lambda), \mathbf{g}(\lambda))$ (solutions of Einstein's equations for every $\lambda \geq 0$) such that $\lambda = 0$ and $\lambda = 1$ correspond to the original and perturbed space-time, respectively, and such that $\mathbf{g}(\lambda)$ is an analytic function of λ in a neighborhood of $\lambda = 0$. Then, we have defined in this form a fiber bundle $\mathcal{M}$ with base $M(\lambda)$ and whose fibers are the manifolds $M(\lambda)$. Now, to compare the several elastic systems obtained by means of any possible perturbation, we need to transform all the manifolds $M(\lambda)$ onto a common one, e.g., $M(0) = M$. This can be done by means of a vector field $\mathbf{v}$ defined on $\mathcal{M}$ that, at any point of $\mathcal{M}$, is not tangent to the corresponding manifold $M(\lambda)$ (Stewart and Walker, 1974). Then, this vector field defines locally a one-parameter family of diffeomorphysms

$$\phi_\lambda: \quad M(0) \to M(\lambda) \qquad (\forall \lambda \geq 0) \tag{12}$$

so that, for a fixed λ, ϕ_λ identifies a generic point of $M(\lambda)$ with a point of $M(0)$, provided that these two points are on the same integral curve of the vector field $\mathbf{v}$. Then, if Q_λ represents an arbitrary tensor field on $M(\lambda)$, and if $\phi_\lambda^* Q_\lambda$ denotes the pullback of Q_λ along ϕ_λ^*, we have in the first order of approximation and for each $x \in M(0)$ that

$$(\phi_\lambda^* Q_\lambda)(x) = Q_0(x) + \lambda(\mathcal{L}_\mathbf{v} Q_\lambda)(x) + O(\lambda^2) \tag{13}$$

where $\lambda \mathcal{L}_\mathbf{v} Q_\lambda$ is the tensor field on M called the first-order perturbation in Q (Schutz, 1987).

If a new transverse vector field $\mathbf{W}$ is chosen on $\mathcal{M}$, another in principle different one-parameter family of diffeomorphysms will be generated. But if the fields $\mathbf{v}$ and $\mathbf{w}$ are related by

$$\mathbf{v}|_0 = \mathbf{w}|_0 + \boldsymbol{\xi} \tag{14}$$

where $\boldsymbol{\xi}$ is a vector field on M (called the displacement vector), then the relationship between the first-order perturbations $\mathcal{L}_v Q$ and $\mathcal{L}_w Q$ is given by the known property of the Lie derivative:

$$\mathcal{L}_\mathbf{v} Q = \mathcal{L}_\mathbf{w} Q + \mathcal{L}_{\boldsymbol{\xi}} Q \tag{15}$$

2.3. Lagrangian Perturbation for the Energy-Momentum Tensor

From now on we will assume that our original material system is static with energy density $\rho(0)$. This means that in the space-time $(M(0), \mathbf{g}(0))$ there is a timelike Killing vector field with spacelike hypersurfaces which are orthogonal to the orbits of the isometry associated to that field. Furthermore, we will suppose the system to be subject to a stress of hydrostatic type,

$$\mathbf{S}(0) = -p\mathcal{P}^{\#} \tag{16}$$

where p is the pressure and the superscript $\#$ represents the operator that acting on a tensor gives the associated contravariant tensor.

On the other hand, we will suppose that in the perturbed state the elastic earth is rotating with a constant angular velocity $\Omega(\lambda)$ so that if L denotes its typical length, we have

$$\varepsilon := \Omega(\lambda) \cdot L \ll 1 \tag{17}$$

Therefore, the gravitational field in the perturbed state is stationary.

If $\mathop{\mathbf{S}}\limits_{\circ}(\lambda)$ denotes the initial stress tensor, then for the perturbed configurations, in which the elastic coefficients of higher order than the first one are neglected, the stress tensor field is given by (Glass and Winicour, 1973)

$$\begin{aligned} \mathbf{S}^b(\lambda) = \mathop{\mathbf{S}}\limits_{\circ}{}^b(\lambda) &+ 4\,\mathrm{sym}[\mathbf{e}(\lambda)\cdot \mathop{\mathbf{S}}\limits_{\circ}(\lambda)]^b \\ &+ C^{34}_{16}[\mathbf{E}(\lambda)\otimes \mathbf{e}(\lambda)]^b - \mathop{\mathbf{S}}\limits_{\circ}[\mathrm{tr}\,\mathbf{e}(\lambda)] + O(\mathbf{e}^2) \end{aligned} \tag{18}$$

where (1) $\mathrm{tr}\,\mathbf{e}(\lambda)$ represents the trace of the relative deformation tensor (11); (2) sym denotes the usual symmetrization operation; and (3) $\mathbf{E}(\lambda)$ is the elasticity tensor field, which is of type $\binom{4}{0}$; it is defined on $M(\lambda)$, and satisfies:

(i) Voight symmetries:

$$\begin{aligned} &\mathbf{E}(\lambda)(\theta^1, \theta^2, \theta^3, \theta^4) \\ &\quad = \mathbf{E}(\lambda)(\theta^2, \theta^1, \theta^3, \theta^4) \\ &\quad = \mathbf{E}(\lambda)(\theta^1, \theta^2, \theta^4, \theta^3) \\ &\quad = \mathbf{E}(\lambda)(\theta^3, \theta^4, \theta^2, \theta^1) \end{aligned} \tag{19}$$

where $\{\theta^i\}$ $(i=1,2,3,4)$ is a base of the cotangent space $T^*M(\lambda)$ at every point $x \in M$.

(ii) Orthogonality with respect to $\mathbf{u}$:

$$\mathbf{E}(\lambda)\cdot \mathbf{u}^b(\lambda) = 0 \tag{20}$$

(iii) There is the limit

$$\lim_{\lambda\to 0} \mathbf{E}(\lambda) =: E \tag{21}$$

If there is defined a world tube ϕ_λ on each manifold $M(\lambda)$, we can define diffeomorphysms ϕ^L_λ of the form

$$\begin{aligned} \phi^L_\lambda:\quad \phi(\mathcal{D}\times\mathbb{R}) \subset M(0) &\to \phi_\lambda(\mathcal{D}\times\mathbb{R}) \subset M(\lambda) \\ \phi(X,t) &\to \phi^L_\lambda(\phi(X,t)) := \phi_\lambda(X,t) \end{aligned} \tag{22}$$

keeping the time order on every world line $\phi_{\lambda X}$ and satisfying

$$\phi^{L*}_\lambda \mathbf{u}(\lambda) = \mathbf{u}(0) \tag{23}$$

and

$$\phi_\lambda^{L*}\underset{\circ}{\mathscr{P}}(\Sigma(\lambda)) = \underset{\circ}{\mathscr{P}}(\Sigma(0)) \tag{24}$$

where $\Sigma(0)$ is a timelike hypersurface on $M(0)$. The orthogonal projector $\underset{\circ}{\mathscr{P}}(\lambda)$ is invariant for the diffeomorphysms ϕ_λ^L, i.e.,

$$\phi_\lambda^{L*}\underset{\circ}{\mathscr{P}}^b(\lambda) = \underset{\circ}{\mathscr{P}}^b(0) \tag{25}$$

Using the classic notation ΔQ to designate the first-order perturbation for an arbitrary quantity Q associated to ϕ_λ^L, which in fact is its Lagrangian variation, we have from (18) (Glass and Winicour, 1973)

$$\begin{aligned}\Delta\mathbf{S} = {} & C_{12}^{34}(\mathbf{E}\otimes\Delta\mathbf{e}) - (\Delta\,\mathrm{tr}\,\mathbf{e})\underset{\circ}{\mathbf{S}} \\ & + 2[\mathrm{sym}(\mathbf{u}\otimes\underset{\circ}{\mathbf{S}})]\cdot\Delta\mathbf{u}^b\end{aligned} \tag{26}$$

where the Lagrangian variation $\Delta\mathbf{e}$ for the deformation tensor (11) is given by

$$\Delta\mathbf{e} = \tfrac{1}{2}\Delta\mathscr{P}^b \tag{27}$$

and the stress tensor $\underset{\circ}{\mathbf{S}}$ by

$$\underset{\circ}{\mathbf{S}} := \underset{\circ}{\mathbf{S}}(0) \tag{28}$$

Finally, taking into account the consistence condition imposed on the equation of state, which let us write the Lagrangian variation of the energy density as

$$\Delta\rho = -\tfrac{1}{2}C_{12}^{12}(\underset{\circ}{\mathbf{S}} + p\mathscr{P}^{\#})\otimes\Delta\mathbf{g} \tag{29}$$

the perturbation for the energy-momentum tensor (4) can be written in the form

$$\begin{aligned}\Delta\mathbf{T} = {} & [-\tfrac{1}{2}C_{12}^{12}(\underset{\circ}{\mathbf{S}} + \rho\mathscr{P}^{\#})\otimes\Delta\mathbf{g}]\mathbf{u}\otimes\mathbf{u} + C_{12}^{34}(\mathbf{E}\otimes\Delta\mathbf{e}) \\ & - \Delta(\mathrm{tr}\,\mathbf{e})\underset{\circ}{\mathbf{S}} + 2C_1^3\{[\mathrm{sym}(\mathbf{u}\otimes\underset{\circ}{\mathbf{S}})]\otimes\Delta\mathbf{u}^b\}\end{aligned} \tag{30}$$

where all the quantities in (30) are defined on $M(0)$ (Carter and Quintana, 1972).

3. EQUATIONS FOR THE PERTURBED GRAVITATIONAL FIELD AND SYMMETRIES

3.1. Successive Approximations

A suitable way to plan in general relativity a problem where the gravitational field is weak lies in: (1) defining the energy-momentum $\mathbf{T}$ for the material system, and the metric $\mathbf{g}$ for the corresponding space-time, onto a (topologically) Euclidean and Minkowskian 4-manifold M endowed with a Cartesian coordinate system $\{x^a\}$ such that, at every point $x \in M$, the metric tensor is the Kronecker delta; (2) taking the weakness of the gravitational field in the sense that the dimensionless parameter k, defined as the ratio of the mass of any body in the system to its typical size, is less than one; (3) seeing that at every point in the interior $\mathscr{A}$ of the world tubes associated to these bodies, the components T^{ab} of the field $\mathbf{T}$ satisfy the conditions

$$T^{ab} = O(k) \tag{31}$$

and (4) having that in the region $\mathscr{B}$ exterior to these world tubes, $\mathbf{T}$ is null; and at the frontier $\mathscr{C}$, the boundary conditions

$$T^{ab} n_b = 0 \tag{32}$$

are satisfied (n_b is the unit Minkowskian normal to $\mathscr{C}$ at every point of $\mathscr{C}$).

Then, for the chosen coordinates, the metric $\mathbf{g}$ has the form

$$g_{ab} = \delta_{ab} + \gamma_{ab} \tag{33}$$

where γ_{ab} are small deviations [i.e., $\gamma_{ab} = O(k)$] with respect to the Minkowskian metric and are subject to the conditions that the signature of $\mathbf{g}$ is preserved.

Using Synge's approximation method (Synge, 1970) up to the nth approximation, we can obtain the expression for g_{ab} as a functional of T^{ab} so that the Einstein equations are satisfied with an error $E^{ab} = O(k^{n+1})$. In this method the Einstein equations are relaxed and substituted by the recurrence formula

$$\underset{m}{L_{ab}} = (-\underset{m-1}{H^{ab}} + JD^{ab}_{rs}\underset{m-1}{H^{rs}}) \qquad (m = 1, \ldots, n) \tag{34}$$

where: (1) $\underset{m}{L_{ab}}$ is the part of the Einstein tensor G^{ab} linear in γ_{ab}, given by

$$\begin{aligned} \underset{m}{L_{ab}} = \tfrac{1}{2}(&\underset{m}{\gamma_{ab,cc}} + \underset{m}{\gamma_{cc,ab}} - \underset{m}{\gamma_{ac,cb}} - \underset{m}{\gamma_{bc,ca}}) \\ &-\tfrac{1}{2}\delta_{ab}(\underset{m}{\gamma_{ab,dd}} - \underset{m}{\gamma_{cd,cd}}) \end{aligned} \tag{35}$$

(2) $\underset{m-1}{H^{ab}}$ is defined by

$$\underset{m-1}{H^{ab}} := T^{ab} + \kappa^{-1}\underset{m-1}{\hat{G}^{ab}}, \qquad (m = 1, \ldots, n; \quad \kappa = 8\pi) \tag{36}$$

$\hat{G}^{ab}$ is the nonlinear part of γ; (3) J is the inverse d'Alembert integral operator in the Minkowskian space-time (i.e., $\square^{-1} = J$); and (4) D^{ab}_{rs} is the differential operator given by

$$D^{ab}_{rs} := \delta_{ar}\frac{\partial^2}{\partial x^b\,\partial x^s} + \delta_{bs}\frac{\partial^2}{\partial x^a\,\partial x^r} - \delta_{ar}\frac{\partial^2}{\partial x^r\,\partial x^b} \tag{37}$$

If, once the approximation method is started, the origin of the coordinate system is chosen so that the following coordinate conditions are required to be satisfied

$$\underset{m}{\gamma^*_{ab,b}} = 0 \tag{38}$$

where

$$\gamma^*_{ab} := \gamma_{ab} - \tfrac{1}{2}\delta_{ab}\gamma_{cc} \tag{38'}$$

then (34) takes the form

$$\tfrac{1}{2}\Box\,\underset{m}{\gamma^*_{ab}} = -\kappa(-\underset{m-1}{H^{ab}} + JD^{ab}_{rs}\underset{m-1}{H^{rs}}) \tag{39}$$

The solutions of (39) are given in terms of the integrodifferential operator K^{ab}_{rs} defined by

$$K^{ab}_{rs} := -\delta_{ar}\delta_{bs}J + JD^{ab}_{rs}J \tag{40}$$

by means of the recurrence law

$$\underset{0}{\gamma^*_{ab}} = 0, \qquad \underset{m}{\gamma^*_{ab}} = -2\kappa K^{ab}_{rs}\underset{m-1}{H^{rs}} \qquad (m = 1, \ldots, n) \tag{41}$$

so that, if in the nth iteration the tensor H^{ab} satisfies the equation of motion

$$\underset{n-1}{H^{ab}_{,b}} = 0 \tag{42}$$

then the metric deviation in the nth order of approximation, given by

$$\underset{n}{\gamma^*_{ab}} = -2\kappa K^{ab}_{rs}\underset{n-1}{H^{rs}} \tag{43}$$

is the solution for the equation

$$\tfrac{1}{2}\underset{n}{\gamma^*_{ab}} = -\kappa T^{ab} - \underset{n-1}{\hat{G}^{ab}} \tag{44}$$

Calculating the Einstein tensor with the metric

$$g_{ab} = \delta_{ab} + \underset{n}{\gamma_{ab}}$$

with γ_n given in (43), and defining the tensor $\mathcal{T}^{ab}$ by (Pechlaner and Synge, 1968)

$$\mathcal{T}^{ab} := -\kappa^{-1}G^{ab} \tag{45}$$

it results that $(\mathcal{T}^{ab}, g_{ab})$ can be interpreted as an exact solution of the gravitational field equations with the property that (45) is not null in $\mathcal{B}$, although its value there, given by the residual part $E^{ab} = O(k^{n+1})$, can be made arbitrarily small by increasing the order of approximation.

3.2. Equations for the Perturbed Field

Let us consider that $(\mathcal{T}^{ab}(0), g_{ab}(0))$ represents an exact solution of a certain unperturbed problem associated, according to the previous interpretation, with an approximate solution in the nth order of approximation. On every space-time $(M(\lambda), g_{ab}(\lambda))$ corresponding to a perturbed state, we can define quasi-Cartesian coordinates so that the metric tensor has the form

$$g_{ab}(\lambda) = \delta_{ab} + \gamma_{ab}(\lambda) \tag{46}$$

To establish the identification of the manifolds $M(\lambda)$ according to (12), it is more convenient not to use the diffeomorphysms (22), but another instead, say ϕ_λ^E, that relate every point in $M(0)$ to another in $M(\lambda)$ with its same coordinates (x^a). Then, if the first-order perturbation (13) associated to ϕ_λ^E for an arbitrary quantity Q is denoted by the classic symbol δQ (Eulerian variation of Q), as δQ is related to ΔQ through (15), we can write the "energy-momentum" tensor for the perturbed state in the form

$$\mathcal{T}^{ab}(\lambda) = T^{ab}(0) + E^{ab}(0) + \delta T^{ab} + \delta E^{ab}(0) \tag{47}$$

Therefore, if the first-order perturbation for the residual part E^{ab} satisfies

$$\delta E^{ab}(\lambda) \leq E^{ab}(\lambda) \qquad (\forall \lambda \geq 0) \tag{48}$$

then, from (47), we have

$$T^{ab}(\lambda) = T^{ab}(0) + \delta T^{ab} + O(k^{n+1}) \tag{49}$$

because $E^{ab}(\lambda) = O(k^{n+1})$.

Now, taking into account that the metric tensors

$$g_{ab}(0) = \delta_{ab} + \underset{n}{\gamma_{ab}}(0) \tag{50}$$

and

$$g_{ab}(\lambda) = \delta_{ab} + \underset{n}{\gamma_{ab}} \tag{50$'$}$$

corresponding to the original and perturbed space-times satisfy exactly the gravitational field equations

$$\underset{n}{L_{ab}}(0) + \underset{n}{\hat{G}^{ab}}(0) = -\kappa \mathcal{T}^{ab}(0) \tag{51}$$

and

$$\underset{n}{L_{ab}}(\lambda) + \underset{n}{\hat{G}^{ab}}(\lambda) = -\kappa \mathcal{T}^{ab} \tag{51$'$}$$

respectively, it results that the metric perturbation associated with the diffeomorphysm ϕ_λ^E, which is given in accordance with (13) by

$$g_{ab}(\lambda) =: g_{ab}(0) + h_{ab} \tag{52}$$

has to satisfy the equation

$$\underset{n}{\delta L_{ab}} + \underset{n}{\delta \hat{G}^{ab}} = -\kappa \delta \mathscr{T}^{ab} \tag{53}$$

or, what is equivalent, the approximate equation

$$\underset{n}{\delta L_{ab}} + \underset{n}{\delta \hat{G}^{ab}} = -\kappa \delta T^{ab} + O(k^{n+1}) \tag{54}$$

Now, in applying the successive approximation method, we will hold the expression for the tensor T^{ab} as fixed, though its "geometric" expression is changing for every order of approximation. We also keep a fixed form both for T^{ab} and δT^{ab}, and we suppose that the perturbations for the energy-momentum and metric tensor are subject to the conditions

$$\delta T^{ab} = O(\varepsilon) \tag{55}$$

and

$$h_{ab} = O(\varepsilon) \tag{56}$$

respectively [ε is the parameter (17) associated with the perturbation]. For (55) to be compatible with (31) we assume that $\varepsilon \leq k$ and we will consider perturbations for which $\varepsilon^2 \leq k^{n+1}$. Then, denoting by m the least natural number such that $k^m \varepsilon \leq k^{m+1}$, (54) can be written in the form

$$\underset{n}{\delta L_{ab}} = -\kappa \delta H^{ab} + O(k^m \varepsilon) \tag{57}$$

where δH^{ab} is the Eulerian variation for (36), that is to say,

$$\delta H^{ab} = \delta T^{ab} + \kappa^{-1} \delta \hat{G}^{ab} \tag{58}$$

Now, as the Eulerian variation for (35), which is given by

$$\begin{aligned} \underset{n}{\delta L_{ab}} &= \tfrac{1}{2}(h_{ab,cc} + h_{cc,ab} - h_{ac,cb} - h_{bc,ac}) \\ &\quad - \tfrac{1}{2}\delta_{ab}(h_{cc,dd} - h_{cd,cd}) + O(\varepsilon^2) \end{aligned} \tag{59}$$

(commas denoting partial differentiation), satisfies the identity

$$\underset{n}{\delta L_{ab,b}} = O(\varepsilon^2) \tag{60}$$

it is clear that the number of functionally independent equations in (57) reduces to six, whereas the number of unknowns (ξ^a, h_{ab}) is 14. It is therefore possible to impose coordinate conditions, and for that we choose them to be of the type

$$h^{*}_{ab,b} = O(\varepsilon^2) \tag{61}$$

Then, taking into account (61), equation (57) becomes

$$h^{*}_{ab} = -2\kappa\delta H^{ab} + O(k^m\varepsilon) \tag{62}$$

3.3. Coordinate Symmetries

Up to this point the reasoning has been carried out without establishing any particular figure for the material system. We will now impose coordinate symmetries compatible with equilibrium figures for an axis-symmetric earth endowed with planes of symmetry as well as with steady rotations for its corresponding perturbed states. To this end we require the tensor fields $g_{ab}(\lambda)$, $T^{ab}(\lambda)$ (with $\lambda \geq 0$), and ξ^a to be invariant under: (1) transformations of the group $\mathcal{G}$ generated by the one-parameter transformations

$$f_1:\quad \{x'^1 = x^1, x'^2 = x^2, x'^3 = x^3, x'^4 = x^4 + \tau\} \tag{63}$$

and

$$f_2:\quad \{x'^1 = x^1\cos\alpha - x^2\sin\alpha, x'^2 = x^1\sin\alpha + x^2\cos\alpha, x'^3 = x^3, x'^4 = x^4\} \tag{64}$$

(where τ and α are real parameters); and (2) the discrete transformations

$$f_3:\quad \{x'^1 = x^1, x'^2 = x^2, x'^3 = -x^3, x'^4 = x^4\} \tag{65}$$

and

$$f_4:\quad \{x'^1 = x^1, x'^2 = x^2, x'^3 = x^3, x'^4 = -x^4\} \tag{66}$$

As is known, the most general symmetric tensor Q of type $\binom{2}{0}$, or $\binom{0}{2}$, $\mathcal{G}$-invariant under transformations of $\mathcal{G}$, has components

$$\begin{pmatrix} [(x^1)^2q_1 + q_2] & x^1x^2q_1 & x^1q_3 & -x^2q_4 \\ & [(x^2)^2q_1 + q_2] & x^2q_3 & x^1q_4 \\ & & q_5 & 0 \\ & \text{sym} & & q_6 \end{pmatrix} \tag{67}$$

where q_i are functions of x^3 and of the radius, defined by

$$r^2 := (x^1)^2 + (x^2)^2 \tag{68}$$

and where q_3 is odd in x^3, and q_1, q_2, q_4, q_5, q_6 are even in x^3. The most general $\mathcal{G}$-invariant coordinate displacement has components

$$\xi^1 = \xi_1 x^1, \qquad \xi^2 = \xi_1 x^2, \qquad \xi^3 = \xi_2, \qquad \xi^4 = 0 \tag{69}$$

where

$$\zeta_j = \zeta_j(r, x^3) \qquad (j = 1, 2) \tag{70}$$

ζ_1 and ζ_2 are even and odd in x^3, respectively. This way, since all the fields considered in the problem will be $\mathscr{G}$-invariant, we reduce the number of unknowns from 14 to 8 in (62).

4. EXPRESSION OF THE EQUATIONS OF MOTION FOR THE PERTURBATION WITH $n=4$

Now, due to the coordinate symmetries imposed on the earth in the previous section, the components of the tensors h^*_{ab} and δH^{ab} in (62) are of the form (67). Then, denoting by $h_i = h_g i(r, x^3)$ and $H_i = H_i(r, x^3, \xi^a)$ $(i=1,\ldots,6)$ the functions q_i in (67) corresponding to h^*_{ab} and δH^{ab}, respectively, a straightforward calculation shows that the coordinate conditions (61) reduces to the following equations:

$$\begin{gathered} rh_{1,r}+3h_1+\frac{1}{r}h_{2,r}+h_{3,x^3}=O(\varepsilon^2) \\ rh_{3,r}+h_3+h_{5,x^3}=O(\varepsilon^2) \end{gathered} \tag{71}$$

On the other hand, taking into account that for a stationary gravitational field such as the one we are considering, the d'Alembert and Laplace operators are coincident, then eqs. (62) that are functionally independent have the final form

$$\begin{gathered} \nabla^2 h_1+\frac{4}{r}h_{1,r}=-2\kappa H_1+O(\varepsilon^2) \\ \nabla^2 h_2+2h_1=2\kappa H_2+O(\varepsilon^2) \\ \nabla^2 h_3+\frac{2}{r}h_{3,r}=-2\kappa H_3+O(\varepsilon^2) \\ \nabla^2 h_4+\frac{2}{r}h_{4,r}=-2\kappa H_4+O(\varepsilon^2) \\ \nabla^2 h_5=-2\kappa H_5+O(\varepsilon^2) \\ \nabla^2 h_6=-2\kappa H_6+O(\varepsilon^2) \end{gathered} \tag{72}$$

The point now is to calculate the functions H_i in the lowest order of approximation that we need for the effects due to the elastic structure in the perturbed state to manifest themselves in the deformation of this earth. To do this, we will calculate the Eulerian variation for the tensor H^{ab}, defined in (36), by deriving, first, the variation for the energy-momentum, δT^{ab}, and second, the corresponding one for the truncated Einstein tensor $\delta\hat{G}^{ab}$. To calculate δT^{ab}, we will suppose, in addition to the assumptions leading to (30), that in the perturbed state the earth has the structure of a solid of isotropic elastic type and therefore that the elasticity tensor $\mathbf{E}$ in

(26) has the following components:

$$E^{abcd} = \lambda \mathcal{P}^{ab}\mathcal{P}^{cd} + 2\mu \mathcal{P}^{a(c}\mathcal{P}^{d)b} \tag{73}$$

where λ and μ are the Lame coefficients. Then, by using (15), (16), (30), and (73), we have for the Eulerian variation of the energy-momentum as a function of the exact metric tensor (33) the following expression:

$$\begin{aligned}\delta T^{ab} = \tfrac{1}{2}\{&\rho \mathcal{P}^{cd} u^a u^b - p[\mathcal{P}^{cd}(u^a u^b + \mathcal{P}^{ab}) - 4\mathcal{P}^{c(b}u^{a)}u^d] \\ &- \lambda \mathcal{P}^{ab}\mathcal{P}^{cd} - 2\mu \mathcal{P}^{a(c}\mathcal{P}^{d)b}\} \\ &\times (h^*_{cd} - \tfrac{1}{2}\delta_{cd} h^{*c}_c + \xi_{c,d} + \xi_{d,c} + 2r^i_{cd}\xi_i) \\ &- (\rho u^a u^b - p\mathcal{P}^{ad})_{,c}\xi^c - 2[\rho u^c u^{(a}\xi^{b)}_{,c} \\ &- p\mathcal{P}^{c(a}\xi^{b)}_{,c}] + O(\varepsilon^2)\end{aligned} \tag{74}$$

In the weak static gravitational field approximation, the orders of magnitude with respect to the parameter k for the functions ρ, u^a, and p in the energy-momentum tensor are in an orthonormal frame (McCrea and O'Brien, 1978; Carter and Quintana, 1977)

$$\rho = O(k), \qquad u^a = O(k^0), \qquad p \sim \lambda \sim \mu = O(k^2) \tag{75}$$

and, since the metric tensor g_{ab} is $O(k^0)$, then we have for the orthogonal projection tensor

$$\mathcal{P}_{ab} = O(k^0) \tag{76}$$

With respect to the unknown quantities in the perturbed state we admit, in addition to (56), that both the Lagrangian displacement as well as its derivatives satisfy

$$\xi^a = O(\varepsilon) \tag{77}$$

It must be said, as was pointed out previously, that in these derivations we only consider perturbations such that if the weak-field approximation for the original problem is carried out with an error $O(k^{n+1})$, then this error is of the same order of magnitude as the error associated with the perturbation. Then, it can be observed from (74) that the prestressed elastic structure of the earth in rotation gives rise to terms of the order $O(k^2\varepsilon)$ and, therefore, we only need to take them into account if the weak-field approximation is carried out at least in the fourth order of approximation with respect to k [i.e., with an error $O(k^5)$]. In other words, with a lower approximation we would only take into account the perturbative effects due to the rotation. Since it is necessary to consider such an apparently high order to maintain the consistency of the different approximations, the fact is that we only need the metric tensor in the second iteration in the expression (74). This iteration corresponds to the fourth approximation in Synge's method and only contains the first-order terms in k given in expression (A1) of the Appendix.

Now, to complete the calculation for the perturbation δH^{ab}, we will derive the perturbation corresponding to the truncated Einstein tensor. This tensor can be written in the form

$$\hat{G}^{ab} = \mathrm{I}^{ab} + \mathrm{II}^{ab} + \mathrm{III}^{ab} + O(k^5) \tag{78}$$

where I^{ab}, II^{ab}, and III^{ab} are homogeneous functions of the second, third, and fourth order in k, respectively, that depend upon the metric deviation. From (56), together with the relation established between the parameters ε and k when $n=4$, we deduce that the order of magnitude with respect to k for the metric perturbation is such that the Eulerian variation for (78) is given by

$$\delta\hat{G}^{ab} = \delta\mathrm{I}^{ab} + \delta\mathrm{II}^{ab} + O(k^5) \tag{79}$$

In the expressions of $\delta\mathrm{I}^{ab}$ and $\delta\mathrm{II}^{ab}$ we only need the unperturbed metric of the second and first order in k, respectively. Therefore, the tensor given by McCrea (1981) with an error $O(k^4)$ is enough to obtain the Eulerian variation of I^{ab}. Thus, we obtain [see (A5)]

$$\begin{aligned}\delta\mathrm{I}^{ab} = {} & \delta M_{ab} - h_{rs}L_{rabs} - \gamma_{rs}\delta L_{rabs} + \tfrac{1}{2}h_{ab}L^*_{rr} \\ & + \tfrac{1}{2}\gamma_{ab}\delta L^*_{rr} + \delta_{ab}h_{rs}L^*_{rs} + \delta_{ab}\gamma_{rs}\delta L^*_{rs} \\ & - h_{ar}L^*_{rb} + \gamma_{ar}\delta L^*_{rb} + h_{br}L^*_{ra} + \gamma_{br}\delta L^*_{ra}\end{aligned} \tag{80}$$

and for the variation of II^{ab} we have

$$\begin{aligned}\delta\mathrm{II}^{ab} = {} & h_{mn}[rb, m][ra, n] + \gamma_{mn}(\delta[rb, m])[ra, n] \\ & + \gamma_{mn}[rb, m](\delta[ra, n]) - \tfrac{1}{2}\delta_{ab}\{h_{mn}[rs, m][rs, n] \\ & + \gamma_{mn}(\delta[rs, m])[rs, n] + \gamma_{mn}[rs, m](\delta[rs, n])\} \\ & - h_{rs}M_{rabs} - \gamma_{rs}(\delta M_{rabs}) + h_{rp}\gamma_{ps}L_{rabs} \\ & + \gamma_{rp}h_{ps}L_{rabs} + \gamma_{rp}\gamma_{ps}(\delta L_{rabs}) + \delta_{ab}[h_{rs}M^*_{rs} \\ & + \gamma_{rs}\delta M^*_{rs} - h_{rp}\gamma_{ps}L^*_{rs} - \gamma_{rp}h_{ps}L^*_{rs} - \gamma_{rp}\gamma_{ps}\delta L^*_{rs} \\ & - \tfrac{1}{2}h_{rs}\gamma_{pq}L_{rpqs} - \tfrac{1}{2}\gamma_{rs}h_{pq}L_{rpqs} - \tfrac{1}{2}\gamma_{rs}\gamma_{pq}(\delta L_{rpqs})] \\ & - h_{ab}(\tfrac{1}{2}M^*_{pp} + \gamma_{rs}L^*_{rs}) - \gamma_{ab}(\tfrac{1}{2}\delta M^*_{pp} + h_{rs}L^*_{rs} \\ & + \gamma_{rs}\delta L^*_{rs}) - h_{br}M_{ra} + \gamma_{br}(\delta M_{ra}) + h_{ar}M_{rb} \\ & + \gamma_{ar}(\delta M_{rb}) + h_{rs}(\gamma_{ap}L_{rbps} + \gamma_{bp}L_{raps}) \\ & + \gamma_{rs}[h_{ap}L_{rbps} + \gamma_{ap}(\delta L_{rbps}) + h_{bp}L_{raps} \\ & + \gamma_{pb}\delta L_{raps}] - \tfrac{1}{2}h_{ar}\gamma_{bs}L^*_{rr} - \tfrac{1}{2}\gamma_{ar}h_{bs}L^*_{rr} \\ & - \tfrac{1}{2}\gamma_{ar}\gamma_{bs}(\delta L^*_{rr}) + (h_{ar}\gamma_{bs} + \gamma_{ar}h_{bs})L^*_{rs} \\ & + \gamma_{ar}\gamma_{bs}(\delta L^*_{rs}) + h_{rs}(\gamma_{ar}L^*_{bs} + \gamma_{br}L^*_{as}) \\ & + \gamma_{rs}[h_{ar}L^*_{bs} + \gamma_{ar}(\delta L^*_{bs}) + h_{br}L^*_{as} + \gamma_{br}\delta L^*_{ab}] + O(\varepsilon^2)\end{aligned} \tag{81}$$

where the symbols δL_{abcd}, δL_{ab}, $\delta[ab, c]$, δM_{abcd}, and δM_{ab} appearing in (80) and (81) are obtained by Eulerian deviations of (A8)-(A10) in the Appendix. Notice that the metric deviations for the space-time corresponding to the unperturbed problem have to enter in (80) with an error $O(k^3)$, whereas in (81) it is enough to take their expressions with an error $O(k^2)$. Finally, using (74), (80), and (81) in (58), we have the final expression for δH^{ab}, so that the perturbative problem is determined in the approximation required.

APPENDIX

The metric in the second approximation for a static earth is

$$g_{ab} = \delta_{ab} + \underset{1}{\gamma_{ab}} + \underset{2}{\gamma_{ab}} + O(k^3) \tag{A1}$$

with

$$\begin{aligned} g_{\alpha\beta} &= \delta_{\alpha\beta} + (2V\delta_{\alpha\beta}) + [2(-K_{\sigma\sigma} + V^2)\delta_{\alpha\beta} + 4K_{\alpha\beta} + E_{\alpha\beta}] + O(k^3) \\ g_{\alpha 4} &= 0 \\ g_{44} &= 1 + (-2V) + [-2(K_{\sigma\sigma} - V^2)] + O(k^3) \end{aligned} \tag{A2}$$

where

$$\begin{aligned} V &:= -\int T^{44}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-1}\, d_3x' = O(k) \\ K_{\alpha\beta} &:= \int T^{\alpha\beta}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-1}\, d_3x' = O(k^2) \\ E_{\alpha\beta} &:= -\frac{1}{\pi}\int (V_{,\alpha}V_{,\beta} + 2VV_{,\alpha\beta})|\mathbf{x}-\mathbf{x}'|^{-1}\, d_3x' = O(k^2) \end{aligned} \tag{A3}$$

The orthogonal projection contravariant tensor in the first approximation is

$$\mathcal{P}^{ab} = \delta_{ab} - \underset{1}{\gamma_{ab}} + u^a u^b + O(k^2) \tag{A4}$$

The Christoffel symbols of the first and second kind in the first approximation are

$$[ab, c]_1 = \tfrac{1}{2}(\underset{1}{\gamma_{cb,a}} + \underset{1}{\gamma_{ac,b}} - \underset{1}{\gamma_{ab,c}}) \tag{A5}$$

$$\underset{1}{\Gamma^a_{bc}} = \delta_{ab}[bc, d]_1 \tag{A6}$$

The truncated Einstein tensor in the third approximation is

$$\begin{aligned}\hat{G}^{ab} = {} & M_{ab} - \gamma_{rs}L_{rabs} + \tfrac{1}{2}\gamma_{ab}L^*_{rr} + \delta_{ab}\gamma_{rs}L^*_{rs} - 2\gamma_{r(a}L^*_{b)r} \\ & + \gamma_{mn}[rb, m][ra, n] - \tfrac{1}{2}\delta_{ab}\gamma_{mn}[rs, m][rs, n] - \gamma_{rs}M_{rabs} \\ & + \gamma_{rp}\gamma_{ps}L_{rabs} + \delta_{ab}(\gamma_{rs}M^*_{rs} - \gamma_{rp}\gamma_{ps}L^*_{rs}) \\ & - \tfrac{1}{2}\delta_{ab}\gamma_{rs}\gamma_{pq}L_{rpqs} - \gamma_{ab}(\tfrac{1}{2}M^*_{pp} + \gamma_{rs}L^*_{rs}) \\ & - 2M_{r(a}\gamma_{b)r} + 2\gamma_{rs}\gamma_{p(a}L_{b)rps} - \tfrac{1}{2}\gamma_{ab}\gamma_{bs}L^*_{rr} \\ & + \gamma_{ar}\gamma_{bs}L^*_{rs} + 2\gamma_{rs}\gamma_{r(a}L^*_{b)s} + O(k^4) \end{aligned} \tag{A7}$$

where the parentheses enclosing indexes denote symmetrization, and L_{abcd}, L^*_{bc}, M_{abcd}, and M^*_{bc} are defined by

$$L_{abcd} := \tfrac{1}{2}(\gamma_{ad,bc} + \gamma_{bc,ad} - \gamma_{ac,bd} - \gamma_{bd,ac}) \tag{A8}$$

$$L^*_{bc} := L_{mbcm} \tag{A9}$$

$$M_{abcd} := [ad, m][bc, m] - [ac, m][bd, m] \tag{A10}$$

$$M^*_{bc} := M_{mbcm} \tag{A11}$$

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